Time-dynamic Properties of Random Vectors with Conditionally Independent Components

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Abstract

In this work we study several properties of the multivariate hazard rate and mean residual life of random vectors with conditionally independent components. In this work we consider the time-dynamic approach for the definition of the multivariate hazard and mean residual life. In particular we give conditions on two random vectors with the property of being conditionally independent, to compare these two random vectors in the hazard rate order and the mean residual life order. As a consequence we obtain conditions for two notions of dependence, known as the HIF (hazard increasing upon failures) and MRL-DF (mean residual lives decreasing upon failures).

1 Introduction

We consider n nonnegative random variables $T_1, T_2, \ldots, T_n$, which represent the random lifetimes of n units, such that they are conditionally independent given some random vector $\mathbf{\Theta}$, where $\mathbf{\Theta}$ takes values on some set $\chi \subseteq \mathbb{R}^m$. The random vector $\mathbf{\Theta}$ can be interpreted as the random environment, in which the units are working. If $F_i(\cdot|\boldsymbol{\theta})$ denotes the marginal distribution of $T_i(\boldsymbol{\theta}) \equiv (T_i|\mathbf{\Theta} = \boldsymbol{\theta})$, and $G(\boldsymbol{\theta})$ denotes the joint distribution function of $\mathbf{\Theta}$, then the joint distribution of $(T_1, T_2, \ldots, T_n)$ is given by

$$F(t_1, t_2, ...t_n) = \int_\chi \prod_{i=1}^{n} F_i(t_i|\boldsymbol{\theta}) dG(\boldsymbol{\theta}).$$

This model appears for example in de Finetti (1937), Shaked (1977), Barlow and Mendel (1992), Spizzichino (2001) and Lindley and Singpurwalla (2002).

For this model Shaked and Spizzichino (1998) and Khaledhi and Kochar (2001) study which kind of dependence arises in $(T_1, T_2, \ldots, T_n)$ when the random vector $\mathbf{\Theta}$ is unknown. In this paper we study conditions under which given two random vectors as above, they are ordered according to the hazard rate order and mean residual life order.

2 Multivariate time-dynamic orders

Given two random variables X and Y, with survival functions $\overline{F}$ and $\overline{G}$, respectively, we say that X is less than Y in the hazard rate order ($X \leq_{\rm hr} Y$) if (see Shaked and Shanthikumar (1994))

$$\overline{F}(t)\overline{G}(s) \leq \overline{F}(s)\overline{G}(t) \text{ for all } s \leq t.$$

Given two nonnegative random variables X and Y with hazard rates r and s respectively, then $X \leq_{\rm hr} Y$ if $r(t) \geq s(t)$ for all $t \geq 0$.

In the multivariate case it is possible to provide several extensions. We consider the time-dynamic definition of the multivariate hazard rate order introduced by Shaked and Shanthikumar (1987).

Let us consider a random vector $\mathbf{S} = (S_1, \ldots, S_n)$ where the S_i's can be considered as the lifetimes of n units. For $t \geq 0$ let h_t denotes the list of units which have failed and their failure times. More explicitly, a history h_t will denote

$$h_t = \{\mathbf{S}_I = \mathbf{s}_I, \mathbf{S}_{\overline{I}} > t\mathbf{e}\},$$

where $I = \{i_1, \ldots, i_k\}$ is a subset of $\{1, \ldots, n\}$, $\overline{I}$ is its complement with respect to $\{1, \ldots, n\}$, $\mathbf{S}_I$ will denote the vector formed by the components of $\mathbf{S}$ with index in I and $0 < \mathbf{s}_{i_j} < t$ for all $j = 1, \ldots, k$, and $\mathbf{e}$ denotes the vector of 1's of a proper dimension.

Now we proceed to give the definition of the multivariate hazard rate order. Given the history h_t, as above, let $j \in \overline{I}$, its multivariate conditional hazard rate, at time t, is defined as follows:

$$\eta_j(t|h_t) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P[t < S_j \leq t + \Delta t | h_t].$$

Clearly $\eta_j(t|h_t)$ is the "probability" of instant failure of component j, given the history h_t.

Now let $\mathbf{S}$ and $\mathbf{T}$ be two n-dimensional random vectors with hazard rate functions $\eta(\cdot|\cdot)$ and $\lambda(\cdot|\cdot)$, respectively. We say that $\mathbf{S}$ is less than $\mathbf{T}$ in the multivariate hazard rate order (denoted by $\mathbf{S} \leq_{\text{hr}} \mathbf{T}$), if, for every $t \geq 0$,

$$\eta_j(t|h_t) \geq \lambda_j(t|h'_t)$$

where

$$h_t = \{\mathbf{S}_{I \cup J} = \mathbf{s}_{I \cup J}, \mathbf{S}_{\overline{I \cup J}} > t\mathbf{e}\} \tag{1}$$

and

$$h'_t = \{\mathbf{T}_I = \mathbf{t}_I, \mathbf{T}_{\overline{I}} > t\mathbf{e}\} \tag{2}$$

whenever $I \cap J = \emptyset$, $\mathbf{0} \leq \mathbf{s}_I \leq \mathbf{t}_I \leq u\mathbf{e}$, and $\mathbf{0} \leq \mathbf{s}_J \leq u\mathbf{e}$, and $j \in \overline{I \cup J}$.

Given two histories as above, we say that h_t is more severe than h'_t.

The multivariate hazard, as the multivariate likelihood ratio order, is not necessarily reflexive. In fact if the random vector $\mathbf{S}$ satisfies $\mathbf{S} \leq_{\text{hr}} \mathbf{S}$, then it is said to have the HIF property (hazard increasing upon failure, see Shaked and Shanthikumar (1990)), and it can be considered as a positive dependence property. Intuitively the HIF notion means that the failure rates of the surviving components increase with the severeness of their 'past'.

Another stochastic order of interest, from a time-dynamic point of view, is the mean residual order.

Given two random variables X and Y, with mean residual lives

$$m(t) = E[X - t | X > t],$$

and

$$l(t) = E[Y - t | Y > t],$$

respectively, we say that X is less than Y in the mean residual life order ($X \leq_{\text{mrl}} Y$) if (see Shaked and Shanthikumar (1994))

$$m(t) \leq l(t)$$

In the multivariate case, given a n-dimensional random vector $\mathbf{S}$, and a history $h_t = \{\mathbf{S}_I = \mathbf{s}_I, \mathbf{S}_{\overline{I}} > t\mathbf{e}\}$, then for the component $j \in \overline{I}$, its multivariate conditional mean residual function, at time t, is defined as follows:

$$m_j(t|h_t) = E[S_j - t | h_t].$$

In this case $m_j(t|h_t)$ is the expected residual life of component j, given the history h_t.

Now let $\mathbf{S}$ and $\mathbf{T}$ be two n-dimensional random vectors with mean residual life functions $m(\cdot|\cdot)$ and $l(\cdot|\cdot)$, respectively. We say that $\mathbf{S}$ is less than $\mathbf{T}$ in the multivariate mean residual life order (denoted by $\mathbf{S} \leq_{\text{mrl}} \mathbf{T}$), if (see Shaked and Shanthikumar (1991)), for every $t \geq 0$,

$$m_j(t|h_t) \leq l_j(t|h'_t)$$

where h_t and h'_t are given as in (1) and (2), respectively, and $j \in \overline{I \cup J}$.

Again the mrl order is not reflexive, and a random vector $\mathbf{S}$ is said to have the MRL-DF (mean residual lives decreasing upon failure) property (see Shaked and Shanthikumar (1991)). The MRL-DF property is also a positive dependence notion.

3 The results

Let us consider two random vectors $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$ and $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$, where $\mathbf{\Theta}_1$ and $\mathbf{\Theta}_2$ are random vectors of dimension m, where $\mathbf{S}$ and $\mathbf{T}$ are conditionally independent given $(\mathbf{\Theta}_1 = \theta)$ and $(\mathbf{\Theta}_2 = \theta)$, respectively.

Now we state the following result.

Theorem 1. *Let $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$ and $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$ be random vectors as above. If*
i) $(S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta})$ (or $(T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta})$) is decreasing [increasing] in the hazard rate order in $\boldsymbol{\theta}$
ii) $(S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}) \leq_{\mathrm{hr}} (T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta})$, for all $\boldsymbol{\theta}$,
and
iii) $(\mathbf{\Theta}_1|h_t) \geq_{\mathrm{st}} [\leq_{\mathrm{st}}](\mathbf{\Theta}_2|h'_t)$, for every two histories h_t and h'_t, for $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ respectively, where h_t is more severe than h'_t, then

$$(S_1, \ldots, S_n) \leq_{\mathrm{hr}} (T_1, \ldots, T_n).$$

An important consequence of this theorem is the following result.

Theorem 2. *Let $(S_1, \ldots, S_n, \mathbf{\Theta})$ be a random vector as above. If*
i) $(S_i|\mathbf{\Theta} = \boldsymbol{\theta})$ is decreasing [increasing] in the hazard rate order in $\boldsymbol{\theta}$
and
ii) $(\mathbf{\Theta}|h_t) \geq_{\mathrm{st}} [\leq_{\mathrm{st}}](\mathbf{\Theta}|h'_t)$, for every two histories h_t and h'_t, for $(S_1, \ldots, S_n)$, where h_t is more severe than h'_t, then

$$(S_1, \ldots, S_n) \in HIF.$$

Therefore we provide conditions for the HIF property of random vectors with conditionally independent components, which was an open problem in the paper by Shaked and Spizzichino (1998).

Now we give conditions for the mrl order of conditionally independent random vectors.

Theorem 3. *Let $(S_1, \ldots, S_n, \mathbf{\Theta}_1)$ and $(T_1, \ldots, T_n, \mathbf{\Theta}_2)$, be random vectors as above. If*
i) $(S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta})$ (or $(T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta})$) is decreasing [increasing] in the mean residual life order in $\boldsymbol{\theta}$
ii) $(S_i|\mathbf{\Theta}_1 = \boldsymbol{\theta}) \leq_{\mathrm{mrl}} (T_i|\mathbf{\Theta}_2 = \boldsymbol{\theta})$, for all $\boldsymbol{\theta}$,
and
iii) $(\mathbf{\Theta}_1|h_t) \geq_{\mathrm{st}} [\leq_{\mathrm{st}}](\mathbf{\Theta}_2|h'_t)$, for every two histories h_t and h'_t, for $(S_1, \ldots, S_n)$ and $(T_1, \ldots, T_n)$ respectively, where h_t is more severe than h'_t, then

$$(S_1, \ldots, S_n) \leq_{\mathrm{mrl}} (T_1, \ldots, T_n).$$

Now as a consequence we can give a result for the MRL-DF property.

Theorem 4. *Let $(S_1, \ldots, S_n, \mathbf{\Theta})$ be a random vector as above. If*
i) $(S_i|\mathbf{\Theta} = \boldsymbol{\theta})$ is decreasing [increasing] in the mean residual life order in $\boldsymbol{\theta}$
and
ii) $(\mathbf{\Theta}|h_t) \geq_{\mathrm{st}} [\leq_{\mathrm{st}}](\mathbf{\Theta}|h'_t)$, for every two histories h_t and h'_t, for $(S_1, \ldots, S_n)$, where h_t is more severe than h'_t, then

$$(S_1, \ldots, S_n) \in MRL - DF.$$

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